

DAVENPORT CONSTANT FOR COMMUTATIVE RINGS

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ABSTRACT. The Davenport constant is one measure for how “large” a finite abelian group is. In particular, the Davenport constant of an abelian group is the smallest k such that any sequence of length k is reducible. This definition extends naturally to commutative semigroups, and has been studied in certain finite commutative rings. In this paper, we give an exact formula for the Davenport constant of a general commutative ring in terms of its unit group.

1. INTRODUCTION

The Davenport constant is an important concept in additive number theory. In particular, it measures the largest zero-free sequence of an abelian group.

The Davenport constant was introduced by Davenport in 1966 [3], but was actually studied prior to that in 1963 by Rogers [8]. The definition was first extended by Geroldinger and Schneider to abelian semigroups by [4] as follows:

Definition 1.1. For an additive abelian semigroup S , let $d(S)$ denote the smallest $d \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

For any $m \in \mathbb{N}$ and $s_1, \dots, s_m \in S$ there exists a subset $J \subseteq [1, m]$ such that $|J| \leq d$ and

$$\sum_{j=1}^m s_j = \sum_{j \in J} s_j.$$

In addition, [4] showed the following:

Proposition 1.1. *If $|S| < \infty$, then $d(S) < \infty$.*

Wang and Guo [11] then gave the definition of the large Davenport constant in terms of reducible and irreducible sequences, as follows:

Definition 1.2. Let S be a commutative semigroup (not necessarily finite). Let A be a sequence of elements in S . We say that A is *reducible* if there exists a proper subsequence $B \subsetneq A$ such that the sum of the elements in A is equal to the sum of the elements in B . Otherwise, we say that A is *irreducible*.

Definition 1.3. Let S be a finite commutative semigroup. Define the *Davenport constant* $D(S)$ of S as the smallest $d \in \mathbb{N} \cup \{\infty\}$ such that every sequence S of d elements in S is reducible.

Remark. D and d are related by the equation $D(S) = d(S) + 1$.

Note that if S is an abelian group, being irreducible is equivalent to being zero-sum free, so the definition of the Davenport constant here is equivalent to the classical definition of the Davenport constant for abelian groups.

In all following sections, unless otherwise noted:

- All semigroups are unital and commutative. Furthermore, we will use multiplication notation for semigroups, as opposed to the additive convention used in [4, 11, 10, 12].
- Similarly, rings are unital and commutative.
- Sets represented by the capital letter S are semigroups.
- Sets represented by the capital letter T are ideals in a semigroup.
- Sets represented by the capital letters A, B are sequences of elements in a semigroup. In addition, $\pi(A)$ denotes the product of the elements in A .
- Sets represented by the capital letter R are commutative rings.
- By abuse of notation, when we write $D(R)$, we actually mean $D(S_R)$, where S_R is the semigroup of R under multiplication.
- C_n denotes the cyclic group of order n ; $\mathbb{Z}/n\mathbb{Z}$ denotes the ring with additive group C_n .

2. PREVIOUS RESULTS

So far, this general setting has been studied extensively in the cases where the semigroup is the semigroup of a (commutative) ring under multiplication. The general idea in this class of problems is to show that the Davenport constant of the semigroup of a ring R under multiplication is “close” to the Davenport constant of its group of units $U(R)$. The reason this has been done is that very little is known about the precise value of $D(G)$ when G is an abelian group of high rank, and often the unit group of these commutative rings will be a group of high rank. However, as we will show, if there is a general theorem for the Davenport constant of an abelian group, then by Theorem 3.2, we will also have a general theorem for the Davenport constant of an arbitrary finite commutative ring.

Remark. Since $U(R)$ is a sub-semigroup of R , we clearly have $D(R) \geq D(U(R))$, so we would like to say something about the difference $D(R) - D(U(R))$.

In their paper, Wang and Guo showed the following result: ¹

Theorem 2.1 (Wang, Guo, 2008, [11]). *If $R = \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z}$, where each of the n_1, \dots, n_r are odd. Then $D(R) = D(U(R))$.*

Later, Wang showed the following theorem:

Theorem 2.2 (Wang, 2015, [10]). *Suppose $q > 2$ is a prime power. If $R \neq \mathbb{F}_q[x]$ is a quotient ring of $\mathbb{F}_q[x]$, then $D(R) = D(U(R))$.*

However, Wang left the case $q = 2$ open and gave an instance for which $D(R) \neq D(U(R))$. Later, Zhang, Wang, and Qu gave the following bound when $q = 2$:

Theorem 2.3 (Zhang, Wang, Qu, 2015, [12]). *If $f \in \mathbb{F}_2[x]$ is nonconstant and $R = \mathbb{F}_2[x]/(f)$, then $D(U(R)) \leq D(R) \leq D(U(R)) + \delta_f$, where $\delta_f = \deg[\gcd(f, x(x+1))]$.*

3. SUMMARY OF NEW RESULTS

First, we will show that the converse of Proposition 1.1 holds for rings:

Theorem 3.1. *If R is a commutative ring and $D(R) < \infty$, then $|R| < \infty$.*

¹[11] actually erroneously claimed that for general n_i , $D(R) = D(U(R)) + \#\{1 \leq i \leq r : 2 \nmid n_i\}$. The author has corresponded with the authors of [11] on this matter, and they offer the result given above.

However, the main result of this paper will be to relate $D(R)$ and $D(U(R))$ for arbitrary finite rings R .

As we will see, the reason why $D(R) = D(U(R))$ does not hold in general is closely related to the presence of certain index 2 ideals in R , which we can see immediately in the following exact formula for $D(R)$ in terms of the Davenport constants of certain subgroups of $U(R)$:

Theorem 3.2. *Suppose R is of the form*

$$(\mathbb{Z}/2\mathbb{Z})^{k_1} \times (\mathbb{Z}/4\mathbb{Z})^{k_2} \times (\mathbb{Z}/8\mathbb{Z})^{k_3} \times (\mathbb{F}_2[x]/(x^2))^{k_4} \times R',$$

where R' is a product of local rings not isomorphic to $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/8\mathbb{Z}, \mathbb{F}_2[x]/(x^2)$. Then

$$D(R) = \max_{0 \leq a \leq k_2 + k_4, 0 \leq b \leq k_3} \left[D(U(R') \times C_2^{k_2 + k_4 + 2k_3 - a - 2b}) + k_1 + 2a + 3b \right].$$

Remark. Note that since R is finite, R is Artinian, so there is a unique decomposition of R as a product as a product of local rings ([1], §8). Thus the quantities k_1, k_2, k_3, k_4 are well-defined as functions of R .

As a corollary, we get the following bound on $D(R) - D(U(R))$:

Corollary 3.1. *Suppose R is of the form*

$$(\mathbb{Z}/2\mathbb{Z})^{k_1} \times (\mathbb{Z}/4\mathbb{Z})^{k_2} \times (\mathbb{Z}/8\mathbb{Z})^{k_3} \times (\mathbb{F}_2[x]/(x^2))^{k_4} \times R',$$

where R' is a product of local rings not isomorphic to $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/8\mathbb{Z}, \mathbb{F}_2[x]/(x^2)$. Then

$$D(U(R)) + k_1 \leq D(R) \leq D(U(R)) + k_1 + k_2 + k_3 + k_4.$$

In addition, equality holds on the right if $U(R)$ is a power of 2 or if $k_i = 0$ for $i = 2, 3, 4$.

Proof. For the left hand side, note that we can pick $a = b = 0$ to get

$$D(R) \geq D(U(R') \times C_2^{k_2 + k_4 + 2k_3}) + k_1 = D(U(R)) + k_1$$

For the right hand side, we use the well-known facts $D(G \times H) \geq D(G) + D(H) - 1$, and $D(C_2) = 2$ to get

$$\begin{aligned} D(R) &= \max_{0 \leq a \leq k_2, 0 \leq b \leq k_3} \left[D(U(R') \times C_2^{k_2 + k_4 + 2k_3 - a - 2b}) + k_1 + 2a + 3b \right] \\ &= \max_{0 \leq a \leq k_2, 0 \leq b \leq k_3} \left[D(U(R') \times C_2^{k_2 + k_3 + 2k_3 - a - 2b}) + (a + 2b)(D(C_2) - 1) + k_1 + a + b \right] \\ &\leq \max_{0 \leq a \leq k_2, 0 \leq b \leq k_3} \left[D(U(R') \times C_2^{k_2 + k_4 + 2k_3}) + k_1 + a + b \right] \\ &= \max_{0 \leq a \leq k_2, 0 \leq b \leq k_3} \left[D(U(R)) + k_1 + a + b \right] \\ &= D(U(R)) + k_1 + k_2 + k_3 + k_4. \end{aligned}$$

When $k_2 + k_3 + k_4 = 0$, equality clearly holds as the left hand side and the right hand side are the same. On the other hand, when $|U(R)|$ is a power of 2, then $U(R')$ is also a 2-group.

However, by [7], if G and H are 2-groups, then $D(G \times H) = D(G) + D(H) - 1$. Thus

$$\begin{aligned}
D(R) &\geq D(U(R')) + k_1 + 2k_2 + 3k_3 + 2k_4 \\
&= D(U(R')) + (k_2 + k_4 + 2k_3)(D(C_2) - 1) + k_1 + k_2 + k_3 + k_4 \\
&= D(U(R')) \times C_2^{k_2+k_4+2k_3} + k_1 + k_2 + k_3 + k_4 \\
&= D(R) + k_1 + k_2 + k_3 + k_4 \geq D(R),
\end{aligned}$$

so equality holds in this case as well. \square

We also have the following more concise bound on $D(R) - D(U(R))$ that does not depend on writing down a local ring product decomposition.

Corollary 3.2. *Suppose R is a finite commutative ring, and let $n_2(R)$ be the number of index two (prime, maximal) ideals of R . Then $D(R) \leq D(U(R)) + n_2(R)$.*

Proof. Since R is finite (and thus Artinian), R can be expressed as a finite product of local rings. Thus in the setting of Corollary 3.1, it suffices to show that $k_1 + k_2 + k_3 + k_4 \leq n_2(R)$. Since R is Artinian, we have that $R \simeq \prod_{i=1}^n R/\mathfrak{q}_i$, where $\{\sqrt{\mathfrak{q}_i}\}_{i=1}^n = \text{Spec}(R)$. In particular, the number of R/\mathfrak{q}_i that can be isomorphic to $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/8\mathbb{Z}$ or $\mathbb{F}_2[x]/(x^2)$ is at most the number of index 2 ideals in $\text{Spec}(R)$, which is precisely $n_2(R)$. \square

Using Theorem 3.2 and Corollaries 3.1 and 3.2, we will give generalizations of each of the results in the previous section.

The main line of attack to prove Theorem 3.2 will be to reduce the problem to the case of finite local rings and then talk about what happens when we “glue” local rings together via product. However, we will first discuss the gluing mechanism for a more general class of semigroups (“almost unit-stabilized”); this will require the notion of *unit-stabilized pairs* and a *relative Davenport constant*. Afterwards, we will show that finite local rings are almost unit-stabilized, and that more structure holds for all finite local rings other than $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/8\mathbb{Z}$, and $\mathbb{F}_2[x]/(x^2)$.

4. DAVENPORT CONSTANT FOR SEMIGROUPS

Let S be a semigroup and $U(S)$ denote its group of units.

Definition 4.1. We say that a sequence A with elements in S is *reducible* if there exists a proper subsequence $A' \subsetneq A$ such that $\pi(A') = \pi(A)$. Otherwise, A is *irreducible*.

Definition 4.2. The Davenport constant $D(S)$ is the minimum positive integer k such that any sequence A of size k is reducible.

Definition 4.3. For S a semigroup, let $\Delta(S) = D(S) - D(U(S))$.

Remark. Clearly $\Delta(S) \geq 0$.

Lemma 4.1. *Suppose S and S' are semigroups. Then*

$$D(S \times S') \geq D(S) + D(S') - 1.$$

Proof. Let $\ell = D(S) - 1$ and $\ell' = D(S') - 1$. Then there exists an irreducible sequence s_1, \dots, s_ℓ in S of length ℓ and an irreducible sequence $s'_1, \dots, s'_{\ell'}$ in S' of length ℓ' . Then the sequence

$$(s_1, 1_{S'}), \dots, (s_\ell, 1_{S'}), (1_S, s'_1), \dots, (1_S, s'_{\ell'})$$

is an irreducible sequence in $S \times S'$ of length $\ell + \ell' = D(S) + D(S') - 2$, which means that $D(S \times S') \geq D(S) + D(S') - 1$. \square

5. PROOF OF THEOREM 3.1

We will first handle the case where R is infinite. In particular, we will show that if $|R| = \infty$, then $D(R) = \infty$.

Remark. The corresponding theorem for semigroups clearly does not hold: let S be the semigroup with underlying set $\mathbb{Z}^{\geq 0}$ and operation $a * b = \max(a, b)$. Then S is infinite but $D(S) = 2$.

Lemma 5.1. *Suppose G is an infinite abelian group. Then $D(G) = \infty$.*

Proof. Let g_1, \dots, g_n be an irreducible sequence. Then g_1, \dots, g_n, g is not irreducible if and only if g^{-1} is not of the form $g_{i_1} \cdots g_{i_k}$. However, there are at most 2^n such g , which means that we can always extend an irreducible sequence to get a longer irreducible sequence, which means that $D(G) = \infty$. \square

Lemma 5.2. *Suppose R is a ring and $I \subseteq R$ is an ideal. Then $D(R) \geq D(R/I)$.*

Proof. This follows from the fact that any irreducible sequence in R/I can be lifted to an irreducible sequence in R . \square

We are now ready to give the proof of Theorem 3.1:

Theorem. *If R is a commutative ring and $D(R) < \infty$, then $|R| < \infty$.*

Proof. Suppose $D(R) = k < \infty$.

- $|R/\mathfrak{m}|$ is finite for every maximal ideal $\mathfrak{m} \subseteq R$. Otherwise, $U(R/\mathfrak{m})$ is an infinite abelian group, so $D(R/\mathfrak{m}) = \infty$, which means that $D(R) = \infty$ as well.
- R has at most $k - 1$ maximal ideals. Suppose otherwise, that R has k maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_k$. Then by the Chinese Remainder theorem, there exists $a_1, \dots, a_k \in R$ such that $a_i \in \mathfrak{m}_i$ and $a_i - 1 \in \mathfrak{m}_j$ for $i \neq j$, and a_1, \dots, a_k would be an irreducible sequence of length k , contradiction.
- Let $J(R)$ denote the Jacobson radical of R , the intersection of all of the maximal ideals of R . Since R has a finite number of maximal ideals, by the Chinese Remainder Theorem, $R/J(R) \simeq \prod_i R/\mathfrak{m}_i$, which means that $R/J(R)$ is finite.
- On the other hand, $|U(R)| < \infty$, for if $|U(R)| = \infty$, then $D(U(R)) = \infty$, so $D(R) = \infty$ as well.
- Finally, $1 + J(R) \subseteq U(R)$, which means that $|J(R)| < \infty$ as well. Thus $|R| = |R/J(R)| \cdot |J(R)| < \infty$.

\square

6. RELATIVE DAVENPORT CONSTANT

In order to prove Theorem 3.2 (and its subsequent corollaries), we need to introduce the notion of the *relative Davenport constant*, which measures the longest irreducible sequence in S with product lying in an ideal $T \subseteq S$.

Definition 6.1. A set $T \subseteq S$ is an ideal if $S \cdot T \subseteq T$. ($T = \emptyset$ is allowed here.)

Suppose S_1, S_2 are semigroups and $T_1 \subseteq S_1, T_2 \subseteq S_2$ are ideals. Then $T_1 \times T_2 \subseteq S_1 \times S_2$ is also an ideal.

Definition 6.2. Given a semigroup S and an ideal $T \subseteq S$, A is a T -sequence if $\pi(A) \in T$.

Definition 6.3. Given a semigroup S and a (possibly empty) ideal $T \subseteq S$, define the *relative Davenport constant* $D(S, T)$ to be the minimum positive integer k such that any T -sequence A of size k is reducible.

We can again define $d(S, T)$ as the maximum length of an irreducible T -sequence, and we once again have the relation $D(S, T) = d(S, T) + 1$.

Remark. If $T = \emptyset$ is the empty ideal, then $d(S, T)$ is tautologically 0 and so $D(S, T)$ is 1.

Before we proceed, we first need the following helpful lemma.

Lemma 6.1. Let S be a semigroup and T be an ideal. Suppose $s \in S$ and A is a sequence in S of length $D(S, T)$ such that $s \cdot \pi(A) \in T$. Then there exists a proper subsequence $A' \subsetneq A$ such that $s\pi(A') = s \cdot \pi(A)$.

Proof. If $\pi(A) \in T$, then we are done by the definition of $D(S, T)$. Otherwise, consider the sequence $B = A \cup \{s\}$. Since $|B| = D(S, T) + 1 > D(S, T)$, there exists a proper subsequence $B' \subsetneq B$ such that $\pi(B') = \pi(B) = s \cdot \pi(A)$. Note that if $s \notin B'$, then $B' \subseteq A$, which means that $\pi(B') \notin T$ (as $\pi(A) \notin T$ by definition), whereas $\pi(B) \in T$, contradiction. Thus $s \in B'$, so if we let $A' = B' \setminus \{s\}$, then $s \cdot \pi(A') = \pi(B') = \pi(B) = s \cdot \pi(A)$, as desired. \square

Lemma 6.2. Suppose G is an abelian group and H is a subgroup of G , and suppose S is a semigroup. Then

$$D(G \times S, G \times T) \geq D((G/H) \times S, (G/H) \times T) + D(H) - 1.$$

Proof. Let $\ell_1 = D((G/H) \times S, (G/H) \times T) - 1$, $\ell_2 = D(H) - 1$. Then there exists an irreducible $((G/H) \times T)$ -sequence $(g_1, s_1), \dots, (g_{\ell_1}, s_{\ell_1})$ in $(G/H) \times S$ of length ℓ_1 , and an irreducible sequence h_1, \dots, h_{ℓ_2} in H of length ℓ_2 . In this case, if we let g'_i be any lift of g_i from G/H to G , then the sequence

$$(g'_1, s_1), \dots, (g'_{\ell_1}, s_{\ell_1}), (h_1, 1_S), \dots, (h_{\ell_2}, 1_S)$$

is an irreducible $G \times T$ -sequence in $G \times S$ of length

$$\ell_1 + \ell_2 = D((G/H) \times S, (G/H) \times T) + D(H) - 2.$$

\square

7. UNIT-STABILIZED PAIRS

Definition 7.1. An ordered pair (S, T) of a semigroup S and an ideal $T \subseteq S$ is a *unit-stabilized pair* if for all $a, b \in S$ such that $ab \notin T$ and $\text{Stab}_{U(S)}(a) = \text{Stab}_{U(S)}(ab)$, then $ab = au$ for some $u \in U(S)$.

Definition 7.2. We say that S is a *unit-stabilized semigroup* if (S, \emptyset) is a unit-stabilized pair.

Example 7.1. If R is a finite field other than \mathbb{F}_2 , then [the semigroup of] R [under multiplication] is unit-stabilized. $(\mathbb{F}_2, 0)$ is a unit-stabilized pair.

It turns out most local rings R are unit-stabilized. Unit-stabilized rings are nice because they satisfy $\Delta(R) = 0$ and are closed under the operation of taking products. However, if R has residue field \mathbb{F}_2 , then it is in fact not unit-stabilized: $U(R) = 1 + \mathfrak{m}$, where \mathfrak{m} is the maximal ideal of R . Thus if $x \in R$ satisfies $\text{Ann}_R(x) = \mathfrak{m}$, then $\text{Stab}_{U(R)}(x) = \text{Stab}_{U(R)}(0) = U(R)$, but clearly $0 \neq ux$ for any unit u .

As we can see, the unit-stabilized behavior breaks down around 0. However, even when R is not unit-stabilized, it turns out that $(R, 0)$ will be a unit-stabilized pair. We have the following formula that helps for working with unit-stabilized pairs:

Theorem 7.1. *Suppose (S, T) is a unit-stabilized pair and S' is a semigroup and T' is an ideal in S' . Then*

$$D(S \times S', S \times T') = \max[D(U(S) \times S', U(S) \times T'), D(S \times S', T \times T')].$$

Proof. First off, it is clear that $D(S \times S', S \times T') \geq D(U(S) \times S', U(S) \times T')$ and $D(S \times S', S \times T') \geq D(S \times S', T \times T')$, so it suffices to show the inequality in the other direction.

Let

$$\ell = \max[D(U(S) \times S', U(S) \times T'), D(S \times S', T \times T')].$$

Suppose we have a sequence $A = \{a_1, \dots, a_\ell\}$ with $\pi(A) \in S \times T'$, and suppose for the sake of contradiction that A is irreducible.

Let P_S and $P_{S'}$ denote the projection maps from $S \times S'$ to S and S' , respectively. If $P_S(\pi(A)) \in T$, then $\pi(A) \in T \times T'$, and by the definition of $D(S \times S', T \times T')$ we automatically have that A is not irreducible. Thus $P_S(\pi(A)) \notin T$.

Let B be a minimal subsequence of A such that

$$\text{Stab}_{U(S)}(P_S(\pi(B))) = \text{Stab}_{U(S)}(P_S(\pi(A))),$$

and without loss of generality suppose that $B = \{a_1, \dots, a_k\}$. Let $p_0 = (1_S, 1_{S'})$, and for $1 \leq i \leq \ell$, $p_i = a_i p_{i-1}$ and let $K_i = \text{Stab}_{U(S)}(P_S(p_i))$. Then we have

$$0 = K_0 \subseteq K_1 \subseteq \dots \subseteq K_\ell.$$

In addition, for $1 \leq i \leq k$, $K_{i-1} \neq K_i$, so by Lemma 6.2,

$$D(K_i) \geq D(K_{i-1}) + D(K_i/K_{i-1}) - 1 \geq D(K_{i-1}) + 1,$$

which means that $D(K_k) \geq k + 1$. Applying Lemma 6.2 again, we have,

$$D((U(S)/K_k) \times S', (U(S)/K_k) \times T') \leq D(U(S) \times S', U(S) \times T') - D(K_k) + 1 \leq \ell - k.$$

In addition, for $k < i \leq \ell$,

$$\text{Stab}(P_S(p_k)) \subseteq \text{Stab}(P_S(a_i p_k)) \subseteq \text{Stab}(P_S(p_\ell)) = \text{Stab}(P_S(p_k)),$$

so the inclusions are equalities. Since $p_\ell = P_S(\pi(A)) \notin T$ and (S, T) is unit-stabilized, for each $k < i \leq \ell$ there exists a $u_i \in U(S)$ such that $u_i P_S(p_k) = a_i P_S(p_k)$ for $k < i \leq \ell$. Let $t_i = (u_i, P_{S'}(a_i))$. We have $p_k t_i = p_k a_i$ for $k \leq i \leq \ell$. Since t_{k+1}, \dots, t_ℓ is a sequence of length

$$\ell - k \geq D((U(S)/K_k) \times S', (U(S)/K_k) \times T')$$

and $(1, \pi_{S'}(p_k)) \cdot t_{k+1} \cdots t_\ell \in U(S) \times T'$, by Lemma 6.1, there exists a (possibly empty) proper subsequence i_1, \dots, i_m such that $(1, \pi_{S'}(p_k)) \cdot t_{i_1} \cdots t_{i_m} \cdot (u, 1_{S'}) = (1, \pi_{S'}(p_k)) \cdot t_{k+1} \cdots t_\ell$ for

some $u \in K_k$. Then since $u \in \text{Stab}(P_S(p_k))$, $(u, 1) \cdot p_k = p_k$, so

$$\begin{aligned}
a_1 a_2 \cdots a_k \cdot a_{i_1} \cdots a_{i_m} &= p_k \cdot a_{i_1} \cdots a_{i_m} \\
&= p_k \cdot t_{i_1} \cdots t_{i_m} \\
&= (1, P_{S'}(p_k)) \cdot (P_S(p_k), 1) \cdot t_{i_1} \cdots t_{i_m} \\
&= (1, P_{S'}(p_k)) \cdot (P_S(p_k), 1) \cdot (u, 1) \cdot t_{i_1} \cdots t_{i_m} \\
&= (1, P_{S'}(p_k)) \cdot (P_S(p_k), 1) \cdot t_{k+1} \cdots t_\ell \\
&= (1, P_{S'}(p_k)) \cdot (P_S(p_k), 1) \cdot a_{k+1} \cdots a_\ell \\
&= p_k \cdot a_{k+1} \cdots a_\ell \\
&= a_1 \cdots a_\ell = \pi(A).
\end{aligned}$$

This contradicts the assumption that A was irreducible. Thus $D(S \times S', T \times T') \leq \max[D(U(S) \times S', U(S) \times T'), D(S \times S', T \times T')]$, as desired. \square

Corollary 7.1. *Suppose (S, T) is a unit-stabilized pair. Then $D(S) = \max[D(U(S)), D(S, T)]$.*

Proof. Apply Theorem 7.1 to the case where S' is the trivial group and $T' = S'$. \square

Corollary 7.2. *If S is a unit-stabilized semigroup, then $D(S) = D(U(S))$.*

8. ALMOST UNIT-STABILIZED SEMIGROUPS

Lemma 8.1. *If S has a 0 element, then for all semigroups S' ,*

$$D(S \times S', \{0\} \times T') = D(S', T') + D(S, \{0\}) - 1.$$

Proof. First we will show that $D(S \times S', \{0\} \times T) \leq D(S', T') + D(S, \{0\}) - 1$.

Let $\ell = D(S', T') + D(S, \{0\}) - 1$, and suppose we have an irreducible $\{0\} \times T$ -sequence $(s_1, s'_1), \dots, (s_\ell, s'_\ell)$. Consider the smallest subset such that the second coordinate of the product is equal to 0. By the definition of $D(S, \{0\})$, this subset has size at most $k = D(S, \{0\}) - 1$. Without loss of generality suppose $s_1 \cdots s_k = 0$, and let $p = s'_1 \cdots s'_k$. Then the sequence s'_{k+1}, \dots, s'_ℓ has length $D(S', T')$, so by Lemma 6.1 there is some proper subsequence such that $ps'_{i_1} \cdots s'_{i_r} = ps'_{k+1} \cdots s'_\ell$. Since the first coordinate of both sides must be zero, we have found a proper subsequence with the same product, contradiction.

Now to show the reverse inequality, it suffices to construct an irreducible $(\{0\} \times T')$ -sequence of length $D(S', T') + D(S, \{0\}) - 2$. Let $\ell_1 = D(S', T') - 1$ and $\ell_2 = D(S, \{0\}) - 1$. By definition, there exists an irreducible T' -sequence $s'_1, s'_2, \dots, s'_{\ell_1} \in S'$ and an irreducible $\{0\}$ -sequence $s_1, s_2, \dots, s_{\ell_2}$. Then $(1_S, s'_1), \dots, (1_S, s'_{\ell_1}), (s_1, 1_{S'}), \dots, (s_{\ell_2}, 1_{S'})$ is an irreducible $(\{0\} \times T')$ -sequence of length $\ell_1 + \ell_2 = D(S', T') + D(S, \{0\}) - 2$, as desired. \square

Definition 8.1. A semigroup S is *almost unit-stabilized* if S has a 0 element and $(S, \{0\})$ is a unit-stabilized pair.

Remark. If S is unit-stabilized and contains a 0 element, the S is also almost unit-stabilized.

Lemma 8.2. *Suppose S is almost unit-stabilized, and suppose S' is any other semigroup. Then*

$$D(S \times S') = \max(D(U(S) \times S'), D(S, \{0\}) + D(S') - 1).$$

Proof. This follows directly from Theorem 7.1 and Lemma 8.1. \square

Corollary 8.1. *Suppose S is almost unit-stabilized and $\Delta(S) = 0$. Then for any semigroup S' , $D(S \times S') = D(U(S) \times S')$.*

Proof. If $\Delta(S) = 0$, then $D(S, \{0\}) \leq D(U(S))$. Thus by Lemma 4.1,

$$D(S, \{0\}) + D(S') - 1 \leq D(U(S)) + D(S') - 1 \leq D(U(S) \times S'),$$

which means that by Lemma 8.2,

$$D(S \times S') = \max(D(U(S) \times S'), D(S, \{0\}) + D(S') - 1) = D(U(S) \times S').$$

□

Theorem 8.1. *Suppose S_1, \dots, S_k are almost unit-stabilized semigroups. Suppose further that for $i > \ell$, $\Delta(S_i) = 0$. Then*

$$D(S_1 \times \dots \times S_k) = \max_{I \subseteq [1, \ell]} \left[D \left(\prod_{i \in [1, k] \setminus I} U(S_i) \right) + \sum_{i \in I} (D(S_i, \{0\}) - 1) \right].$$

Proof. First, by ℓ applications of Lemma 8.2, we have that for any semigroup S' ,

$$D(S_1 \times \dots \times S_\ell \times S') = \max_{I \subseteq [1, \ell]} \left[D \left(\left(\prod_{i \in [1, \ell] \setminus I} U(S_i) \right) \times S' \right) + \sum_{i \in I} (D(S_i, \{0\}) - 1) \right].$$

Now if we let $S' = S_{\ell+1} \times \dots \times S_k$ and apply Corollary 8.1 $k - \ell$ times, we have

$$D(S_1 \times \dots \times S_k) = \max_{I \subseteq [1, \ell]} \left[D \left(\prod_{i \in [1, k] \setminus I} U(S_i) \right) + \sum_{i \in I} (D(S_i, \{0\}) - 1) \right].$$

□

Corollary 8.2. *If $T = S_1 \times \dots \times S_k$ is a product of almost unit-stabilized semigroups, then*

$$\Delta(T) \leq \Delta(S_1) + \dots + \Delta(S_k).$$

Proof. Applying Theorem 8.1 to the (trivial) case where $\ell = k$, we have

$$D(T) = \max_{I \subseteq [1, k]} \left[D \left(\prod_{i \in [1, k] \setminus I} U(S_i) \right) + \sum_{i \in I} (D(S_i, \{0\}) - 1) \right].$$

However, using the inequality $D(G) \geq D(G/H) + D(H) - 1$ (a special case of Lemma 6.2 where S is the trivial group), we have that for any $I \subseteq [1, k]$,

$$D \left(\prod_{i \in [1, k] \setminus I} U(S_i) \right) + \sum_{i \in I} (D(U(S_i)) - 1) \leq D \left(\prod_{i \in [1, k]} U(S_i) \right) = D(U(T)).$$

Thus

$$\begin{aligned}
D(T) &= \max_{I \subseteq [1, k]} \left[D \left(\prod_{i \in [1, k] \setminus I} U(S_i) \right) + \sum_{i \in I} (D(S_i, \{0\}) - 1) \right] \\
&\leq \max_{I \subseteq [1, k]} \left[D(U(T)) + \sum_{i \in I} (D(S_i, \{0\}) - D(U(S_i))) \right] \\
&= D(U(T)) + \sum_{i=1}^k \max(0, D(S_i, \{0\}) - D(U(S_i))) \\
&= D(U(T)) + \sum_{i=1}^k \Delta(S_i),
\end{aligned}$$

from which the desired result immediately follows. \square

Remark. It would be nice if it were true in general that $\Delta(S_1 \times S_2) \leq \Delta(S_1) + \Delta(S_2)$. However, this is not true, even when one of S_1 or S_2 is almost unit-stabilized! For an example, let $S_1 = \mathbb{Z}/3\mathbb{Z}$ and let $S_2 = S \times S'$, where $S' = \mathbb{Z}/4\mathbb{Z}$ and $S = \{0, 1, 2, 4\} \subseteq \mathbb{Z}/7\mathbb{Z}$.

We have that the following:

- $U(S_1) \simeq C_2, U(S) \simeq C_3, U(S') \simeq C_2$.
- S_1, S, S' are unit-stabilized.
- $D(S_1) = D(U(S_1)) = 2$.
- $D(S) = D(U(S)) = 3$.
- $D(S') = D(S', \{0\}) = 3, D(U(S')) = 2$.
- $D(S_2) = D(S \times S') = D(U(S) \times U(S')) = \max(D(U(S) \times U(S')), D(U(S)) + D(S', \{0\}) - 1) = \max(6, 3 + 3 - 1) = 6, D(U(S_2)) = D(C_6) = 6$.

However,

$$\begin{aligned}
D(S_1 \times S_2) &= D(S_1 \times S \times S') = D(U(S_1) \times U(S) \times U(S')) \\
&= \max(D(C_6 \times U(S')), D(C_6) + D(S', \{0\}) - 1) \\
&= \max(7, 6 + 3 - 1) = 8.
\end{aligned}$$

On the other hand,

$$D(U(S_1 \times S_2)) = D(U(S_1) \times U(S) \times U(S')) = D(C_6 \times C_2) = 7.$$

Thus $\Delta(S_1) = \Delta(S_2) = 0$ but $\Delta(S_1 \times S_2) = 1$. In fact, it is possible to construct S_1, S_2 such that $\Delta(S_1) = \Delta(S_2) = 0$ and one of S_1, S_2 is almost unit-stabilized, but $\Delta(S_1 \times S_2)$ is arbitrarily large.

9. LOCAL RINGS

In this section, we look at the case where R is a (finite) local ring with maximal ideal \mathfrak{m} and residue field k . It turns out local rings are all either unit-stabilized or almost unit-stabilized.

Lemma 9.1. *For all $a \in R$ there exists a positive integer n such that $a\mathfrak{m}^n = 0$.*

Proof. Since R is finite, the chain of ideals

$$aR \supseteq a\mathfrak{m} \supseteq a\mathfrak{m}^2 \supseteq \dots$$

must stabilize, so $a\mathfrak{m}^n = a\mathfrak{m}^{n+1}$ for some positive integer n . Then by Nakayama's lemma we must have $a\mathfrak{m}^n = 0$. \square

Lemma 9.2. *If $a \neq 0$ and $\text{Ann}_R(a) = \text{Ann}_R(ab)$, then $b \in U(R)$.*

Proof. Suppose instead that $b \notin U(R)$, so $b \in \mathfrak{m}$. Consider the minimum n such that $a\mathfrak{m}^n = 0$. (Clearly $n > 0$.) Then $ab\mathfrak{m}^{n-1} \subseteq a\mathfrak{m}^n = 0$ but $a\mathfrak{m}^{n-1} \neq 0$, which means that $\mathfrak{m}^{n-1} \not\subseteq \text{Ann}_R(a)$ but $\mathfrak{m}^{n-1} \subseteq \text{Ann}_R(ab)$, contradiction. \square

Theorem 9.1. *We have:*

- (1) *If $k \not\cong \mathbb{F}_2$, then R is unit stabilized.*
- (2) *If $k \simeq \mathbb{F}_2$, then R is almost unit-stabilized.*

Proof. Note that for all $a \in R$, $\text{Stab}_{U(R)}(a) = (1 + \text{Ann}_R(a)) \cap U(R)$. In addition, if $a \neq 0$, then $\text{Ann}_R U(R)$ is a proper ideal of R , which means that $\text{Ann}_R U(R) \subseteq \mathfrak{m}$, so $1 + \text{Ann}_R U(R) \subseteq U(R)$, which means that $\text{Stab}_{U(R)}(a) = 1 + \text{Ann}_R(a)$. Thus if $ab \neq 0$ and $\text{Stab}_{U(R)}(a) = \text{Stab}_{U(R)}(ab)$, then $\text{Ann}_R(a) = \text{Ann}_R(ab)$, which by the previous lemma implies $b \in U(R)$.

In the case where $R/\mathfrak{m} \not\cong \mathbb{F}_2$, then we have $1 + \mathfrak{m} \subsetneq U(R)$, so for all $a \neq 0$, $\text{Stab}_{U(R)}(a) \subseteq 1 + \mathfrak{m} \subsetneq U(R) = \text{Stab}_{U(R)}(0)$, which means that R is unit-stabilized. \square

Corollary 9.1. *If $k \not\cong \mathbb{F}_2$, then $\Delta(R) = 0$.*

Theorem 9.2. *Suppose R is a finite local ring. Then $\Delta(R) \leq 1$, and equality holds if and only if R is isomorphic to one of the following rings:*

- $\mathbb{Z}/2\mathbb{Z}$
- $\mathbb{Z}/4\mathbb{Z}$
- $\mathbb{Z}/8\mathbb{Z}$
- $\mathbb{F}_2[x]/(x^2)$

Proof. We have already taken care of the case where $k \not\cong \mathbb{F}_2$. Suppose R is a local ring with residue field \mathbb{F}_2 . Then $|R| = 2^n$ for a positive integer n . We will show the following:

- (1) $D(R, 0) \leq n + 1$.
- (2) If $D(R, 0) = n + 1$, then $\mathfrak{m}^{n-1} \neq 0$.
- (3) $D(U(R)) \geq n$.
- (4) If $D(U(R)) = n$, then $U(R) \simeq C_2^{n-1}$.
- (5) If $n \geq 3$ and $\mathfrak{m}^{n-1} \neq 0$ and $U(R) \simeq C_2^{n-1}$, then $\mathfrak{m} = (2)$ and $R \simeq \mathbb{Z}/2^n\mathbb{Z}$.
- (6) If $n \geq 4$, then $D(U(\mathbb{Z}/2^n\mathbb{Z})) \neq C_2^{n-1}$.
- (7) If $n \leq 2$, then $R \simeq \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}$, or $\mathbb{F}_2[x]/(x^2)$.

To finish the proof, note that since R is almost unit-stabilized, $D(R) = \max(D(U(R)), D(R, 0))$, so from (1) and (2) we immediately get $\Delta(R) \leq 1$. Now suppose $\Delta(R) = 1$. Since $D(R, 0) \leq n + 1$, $D(U(R)) \geq n$, and $\Delta(R) = \max(0, D(R, 0) - D(U(R)))$, equality must hold in both cases. Thus by (2), $\mathfrak{m}^{n-1} \neq 0$; by (4), $U(R) \simeq C_2^{n-1}$; by (5), this means that either $|R| \leq 4$ or $R \simeq \mathbb{Z}/2^n\mathbb{Z}$. In the first case, by (7), R must be either $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}$, or $\mathbb{F}_2[x]/(x^2)$. Otherwise, by (6), we must have $n = 3$, in which case $R \simeq \mathbb{Z}/8\mathbb{Z}$.

Finally, we have the following values of $D(R)$ and $D(U(R))$ for $R = \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/8\mathbb{Z}, \mathbb{F}_2[x]/(x^2)$

R	$D(R)$	$D(U(R))$
$\mathbb{Z}/2\mathbb{Z}$	2	1
$\mathbb{Z}/4\mathbb{Z}$	3	2
$\mathbb{Z}/8\mathbb{Z}$	4	3
$\mathbb{F}_2[x]/(x^2)$	3	2

In each of these cases, we have $\Delta(R) = 1$.

Here are proofs of the 7 claims:

- (1) Suppose a_1, \dots, a_{n+1} is an irreducible sequence in R with $a_1 \cdots a_{n+1} = 0$. Then none of a_1, \dots, a_{n+1} are in $U(R)$, or else they could be omitted. In addition, $a_1 \cdots a_n \neq 0$. Thus if $p_i = a_1 \cdots a_i$, we have $\text{Stab}_{U(R)}(p_i) \subsetneq \text{Stab}_{U(R)}(p_{i+1})$ for $0 \leq i < n$. In particular, this means that $|\text{Stab}_{U(R)}(p_i)| \geq 2^i$. However, when $i = n$ this gives $2^{i-1} = |U(R)| \geq |\text{Stab}_{U(R)}(p_i)| \geq 2^i$, contradiction.
- (2) Suppose $D(R, 0) = n + 1$, and let a_1, \dots, a_n be an irreducible sequence in R with $a_1 \cdots a_n = 0$. Then none of the a_i are units, so $a_i \in \mathfrak{m}$. In addition, $a_1 \cdots a_{n-1} \neq 0$, which means that $\mathfrak{m}^{n-1} \neq 0$.
- (3) We have that $|U(R)| = 2^{n-1}$. Thus we can write

$$U(R) = C_{2^{e_1}} \times \cdots \times C_{2^{e_r}},$$

where $e_1 + \cdots + e_r = n - 1$. Since R is a 2-group, by [7],

$$D(U(R)) = 1 + \sum_{i=1}^r (D(C_{2^{e_i}}) - 1) = 1 + \sum_{i=1}^r (2^{e_i} - 1) \geq 1 + \sum_{i=1}^r e_i = n.$$

Here we also used the inequality $2^n - 1 \geq n$ for all $n \in \mathbb{Z}$.

- (4) In the above equation, equality holds if and only if each of the e_i is equal to 1, in which case $U(R) \simeq C_2^{n-1}$.
- (5) Since $\mathfrak{m}^{n-1} \neq 0$, by Nakayama's lemma we have that $\mathfrak{m}^i \subsetneq \mathfrak{m}^{i-1}$ for $1 \leq i \leq n$, in which case $2|\mathfrak{m}^i| \leq |\mathfrak{m}^{i-1}|$. Thus we have

$$2^n = |\mathfrak{m}^0| \geq 2|\mathfrak{m}^1| \geq \cdots \geq 2^{n-1}|\mathfrak{m}^{n-1}| \geq 2^{n-1} \cdot 2 = 2^n,$$

so equality holds in each of the above inequalities. In particular, we have $|\mathfrak{m}/\mathfrak{m}^2| = 2$, so

$$\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1,$$

which means that \mathfrak{m} is principal.

Suppose $\mathfrak{m} = (t)$. Then $t - 1$ is a unit, and since $U(R) \simeq C_2^{n-1}$, we must have $(t - 1)^2 = 1$, or $t^2 = 2t$.

I claim that $2 \in \mathfrak{m} \setminus \mathfrak{m}^2$. Suppose otherwise, that $2 \in \mathfrak{m}^2$. Then we have $t^2 = 2t \in \mathfrak{m}^3$, which means that $\mathfrak{m}^2 = (t^2) = \mathfrak{m}^3$. By Nakayama's lemma, this means that $\mathfrak{m}^2 = 0$, which also means $\mathfrak{m}^{n-1} = 0$ as $n \geq 3$, contradiction.

Thus $2 \in \mathfrak{m} \setminus \mathfrak{m}^2$, which means that $2 = ut$ for some unit u , so $\mathfrak{m} = (t) = (2)$. Finally, note that

$$(2^{n-1}) = \mathfrak{m}^{n-1} \neq 0,$$

so $2^{n-1} \neq 0$. Thus in the additive group of R , 1 has order 2^n , which means that we must have $R \simeq \mathbb{Z}/2^n\mathbb{Z}$.

- (6) If $n \geq 4$, then 5 is a unit in $\mathbb{Z}/2^n\mathbb{Z}$, but $3^2 \not\equiv 1 \pmod{2^n}$, so $U(\mathbb{Z}/2^n\mathbb{Z}) \not\simeq C_2^{n-1}$.

- (7) When $n = 1$, there is a unique ring with two elements, $\mathbb{Z}/2\mathbb{Z}$. Now if $n = 2$, $|\mathfrak{m}| = 2$, so $\mathfrak{m} = \{0, x\}$ for some $x \neq 0, 1$. Then $R = \{0, 1, x, x - 1\}$. We have two cases:
- $2 \neq 0$. Then we must have $x = 2$ and $x - 1 = 3$, so $R \simeq \mathbb{Z}/4\mathbb{Z}$.
 - $2 = 0$. Then $1 = (x + 1)^2 = x^2 + 1$, so $x^2 = 0$, $(x + 1)^2 = 1$, $x(x + 1) = x$. In this case, $R \simeq \mathbb{F}_2[x]/x^2$.

□

10. PROOF OF THEOREM 3.2

Let $R_1 = \mathbb{Z}/2\mathbb{Z}$, $R_2 = \mathbb{Z}/4\mathbb{Z}$, $R_3 = \mathbb{Z}/8\mathbb{Z}$, $R_4 = \mathbb{F}_2[x]/(x^2)$ denote the “bad” local rings. We have that R is of the form

$$R_1^{k_1} \times R_2^{k_2} \times R_3^{k_3} \times R_4^{k_4} \times R',$$

where R' is a product of local rings not isomorphic to R_1, R_2, R_3 , or R_4 . Then by Theorem 9.2, R' is a product of almost unit-stabilized local rings with $\Delta = 0$. In addition, we have

- $U(R_1) = 0$, $D(R_1) = 1$.
- $U(R_2) \simeq C_2$, $D(R_2) = 2$.
- $U(R_3) \simeq C_2 \times C_2$, $D(R_3) = 3$.
- $U(R_4) \simeq C_2$, $D(R_4) = 2$.

Plugging this all into Theorem 8.1, we have

$$\begin{aligned} D(R) &= \max_{n_i \in [0, k_i]} \left[D \left(C_2^{(k_2 - n_2) + 2(k_3 - n_3) + (k_4 - n_4)} \times U(R') \right) + n_1 + 2n_2 + 3n_3 + 2n_4 \right] \\ &= \max_{n_i \in [0, k_i]} \left[D \left(C_2^{k_2 + k_4 + 2k_3 - (n_2 + n_4) - 2n_3} \times U(R') \right) + k_1 + 2(n_2 + n_4) + 3n_3 \right] \\ &= \max_{0 \leq a \leq k_2 + k_4, 0 \leq b \leq k_3} \left[D \left(U(R') \times C_2^{k_2 + k_4 + 2k_3 - a - 2b} \right) + k_1 + 2a + 3b \right]. \end{aligned}$$

11. A FEW MORE COROLLARIES OF THEOREM 3.2

Theorem 11.1. $D(R) = D(U(R))$ in any of the following scenarios:

- (1) $|R|$ is odd.
- (2) R is an \mathbb{F}_q algebra, where q is a power of 2 other than 2.
- (3) $R = \mathbb{Z}/n\mathbb{Z}$, where $16 \mid n$.
- (4) $R = \mathbb{F}_2[x]/(f)$, where $v_x(f), v_{x+1}(f) \notin \{1, 2\}$.
- (5) R is of the form $\mathbb{Z}/4\mathbb{Z} \times R'$ or $\mathbb{F}_2[x]/(x^2) \times R'$, where $|U(R')| > 1$ is odd and $\Delta(R') = 0$.
- (6) $R = \mathbb{F}_2[x]/(x^2 g)$, where g is a product of (at least one) distinct irreducibles that are not x or $x + 1$.

Proof.

- (1) This follows immediately from Corollary 3.2, as a ring with odd order cannot have any ideals of index 2.
- (2) Note that any quotient ring of R is also an \mathbb{F}_q algebra, which means that any proper ideal has index at least q in R . Thus $n_2(R) = 0$, so by Corollary 3.2, $\Delta(R) = 0$.
- (3) In the setting of Corollary 3.1, it is easy to see that for the ring $R = \mathbb{Z}/n\mathbb{Z}$ with $16 \mid n$, so we have $k_1 = k_2 = k_3 = k_4 = 0$, which means that $\Delta(R) = 0$ as well.

- (4) This is a direct consequence of Theorem 11.2 below, which is in turn a consequence of Corollary 3.1.
- (5) We first need the following lemma:

Lemma 11.1. *Suppose G is a nontrivial group of odd order. Then $D(G \times C_2) > D(G) + 1$.*

Proof. (We will use additive notation for this proof). Let (g_1, \dots, g_ℓ) be an irreducible sequence in G , where $\ell = D(G) - 1$. Note that the map $g \mapsto g + g$ in G is injective, which means that the map $(\times \frac{1}{2})$ is well-defined. Then the sequence

$$(g_1, 0), \dots, (g_{\ell-1}, 0), (\frac{1}{2}g_\ell, 1), (\frac{1}{2}g_\ell, 1), (\frac{1}{2}g_\ell, 1)$$

is an irreducible sequence in $G \times C_2$ of length $\ell + 2 = D(G) + 1$, so $D(G \times C_2) > D(G) + 1$. \square

Since $|U(R')|$ is odd, the decomposition of R' into a product of local rings cannot have any rings isomorphic to $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/8\mathbb{Z}$, or $\mathbb{F}_2[x]/(x^2)$. In addition, since $\Delta(R') = 0$, by Corollary 3.1, this decomposition cannot have any rings isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Thus the setting of Theorem 3.2 applies; plugging in both $R = \mathbb{Z}/4\mathbb{Z} \times R'$ and $\mathbb{F}_2[x]/(x^2) \times R'$ gives

$$D(R) = \max(D(U(R')) + 2, D(U(R') \times C_2))$$

However, by the lemma, $D(U(R') \times C_2) > D(U(R')) + 1$, which means that $D(R) = D(U(R') \times C_2) = D(U(R))$.

- (6) If $f \in \mathbb{F}_2$ is irreducible, then the unit group of $\mathbb{F}_2[x]/(f)$ has odd order (in particular, it has order $2^{\deg f} - 1$.) Thus if g is a product of irreducibles other than $x, x + 1$, then $|U(\mathbb{F}_2[x]/(g))| > 1$ is odd, and by Corollary 3.1, $\Delta(\mathbb{F}_2[x]/(g)) = 0$. Thus this is a special case of (5), with $R' = \mathbb{F}_2[x]/(g)$ and $R \simeq \mathbb{F}_2[x]/(x^2) \times R'$, so $\Delta(R) = 0$. \square

Note that (1) in Theorem 11.1 is a generalization of Theorem 2.1 and (2) is a generalization of Theorem 2.2. In addition, we have the following refinement of Theorem 2.3.

Theorem 11.2. *Let $f \in \mathbb{F}_2[x]$ be nonconstant and $R = \mathbb{F}_2[x]/(f)$. If $f = x^a(x+1)^b g$, where $\gcd(g, x(x+1)) = 1$, then*

$$\delta_{a1} + \delta_{b1} \leq D(R) - D(U(R)) \leq \delta_{a1} + \delta_{b1} + \delta_{a2} + \delta_{b2},$$

where δ_{ij} is the Kronecker δ function.

Proof. Note that if f factors into irreducibles as $f = f_1^{e_1} \cdots f_n^{e_n}$, then the decomposition of $\mathbb{F}_2[x]/(f)$ into products of local rings is

$$\mathbb{F}_2[x]/(f) \simeq \mathbb{F}_2[x]/(f_1^{e_1}) \times \cdots \times \mathbb{F}_2[x]/(f_n^{e_n}).$$

In the setting of Corollary 3.1, we have $k_1 = \delta_{a1} + \delta_{b1}$, $k_2 = k_3 = 0$, and $k_4 = \delta_{a2} + \delta_{b2}$. The desired result immediately follows. \square

Finally, we give a partial answer to the general problem stated by Wang and Guo in [11]:

Theorem 11.3. *If $R = \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z}$, then*

$$\#\{1 \leq i \leq r : 2 \parallel n_i\} \leq D(R) - D(U(R)) \leq \#\{1 \leq i \leq r : 2 \mid n_i, 16 \nmid n_i\}.$$

In addition, equality holds on the right when all of the n_i are powers of 2.

Proof. Again in the setting of Corollary 3.1, we have $k_4 = 0$, and for $1 \leq i \leq 3$,

$$k_i = \#\{1 \leq i \leq r : 2^i \parallel n_i\}$$

Thus by Corollary 3.1,

$$D(R) - D(U(R)) \geq k_1 = \#\{1 \leq i \leq r : 2 \parallel n_i\}$$

and

$$\begin{aligned} D(R) - D(U(R)) &\leq k_1 + k_2 + k_3 + k_4 \\ &= \sum_{i=1}^3 \#\{1 \leq i \leq r : 2^i \parallel n_i\} \\ &= \#\{1 \leq i \leq r : 2 \mid n_i, 16 \nmid n_i\}. \end{aligned}$$

In addition, if each of the n_i is a power of 2, $U(R)$ is a 2-group, so by Corollary 3.1, equality holds on the right. \square

12. FURTHER DIRECTION

Unfortunately, we still do not have a complete classification of the finite rings R for which $D(R) = D(U(R))$, even if we restrict to the simple cases where R is of the form $\mathbb{Z}/n\mathbb{Z}$ or $\mathbb{F}_2[x]/(f)$.

As we can see, these questions depend on the relation between $D(G)$ and $D(G \times C_2^e)$ for an abelian group G . This is what we know:

If $G = C_{n_1} \times \cdots \times C_{n_r}$, where $n_1 \mid \cdots \mid n_r$, then we can define $M(G) = 1 + \sum_{i=1}^r (n_i - 1)$. Clearly $D(G) \geq M(G)$. However, we have the following:

Proposition 12.1. *$D(G) = M(G)$ in each of the following cases:*

- (1) G is a p -group. [7]
- (2) $G = C_n \times C_{kn}$. [7]
- (3) $G = C_{p^a k} \times H$, where H is p -group and $p^a \geq M(H)$. [9]
- (4) $G = C_2 \times C_{2n} \times C_{2kn}$, where n, k are odd and the largest prime dividing n is less than 11. [9]
- (5) $G = C_2^3 \times C_{2n}$, where n is odd. [2]

(For a more complete list, see [6].) On the other hand, it is not true that $D(G) = M(G)$ for all G .

Proposition 12.2. *$D(G) > M(G)$ if $G = C_2^e \times C_{2n} \times C_{2nk}$ for n, k odd and $e \geq 3$. [5]*

In what follows, we will try and compute exact values of $\Delta(R)$ when R is of the form $\mathbb{Z}/n\mathbb{Z}$ or $\mathbb{F}_2[x]/(f)$.

12.1. $R = \mathbb{Z}/n\mathbb{Z}$. By Corollary 3.2, we have $\Delta(\mathbb{Z}/n\mathbb{Z}) \leq 1$.

Question. For $n \in \mathbb{Z}^+$, when is $\Delta(\mathbb{Z}/n\mathbb{Z})$ equal to 0 and when is it 1?

Note that we have already resolved this problem for most n . In particular, we have the following:

- If n is odd or $16 \mid n$, $\Delta(R) = 0$.
- If $n = 2b$ where b is odd, then $\Delta(R) = 1$.
- If $n = 4b$ where b is odd, then by Theorem 3.2 we have

$$\Delta(R) = \max(0, D(U(\mathbb{Z}/b\mathbb{Z})) + 2 - D(U(\mathbb{Z}/n\mathbb{Z})))$$

However, $U(\mathbb{Z}/n\mathbb{Z}) = U(\mathbb{Z}/b\mathbb{Z}) \times C_2$, which means that

$$D(U(\mathbb{Z}/n\mathbb{Z})) \geq D(U(\mathbb{Z}/b\mathbb{Z})) + 1.$$

Thus $\Delta(\mathbb{Z}/n\mathbb{Z}) = 1$ if and only if

$$D(U(\mathbb{Z}/n\mathbb{Z})) = D(U(\mathbb{Z}/b\mathbb{Z})) + 1.$$

Using this, we can calculate $\Delta(R)$ in the following special cases using Propositions 12.1 and 12.2 in conjunction with Theorem 3.2:

- If b is a prime power then $\Delta(R) = 1$.
- If b is a product of distinct primes of the form $2^{2^n} + 1$, then $\Delta(R) = 1$.
- If $b = p_1^{e_1} p_2^{e_2}$ and $\gcd(\phi(p_1^{e_1}), \phi(p_2^{e_2}))$ has no prime factors greater than 7, then $\Delta(R) = 1$.
- If $b = p_1^{e_1} p_2^{e_2} p_3^{e_3}$ and $\phi(p_1^{e_1})/2, \phi(p_2^{e_2})/2, \phi(p_3^{e_3})/2$ are pairwise relatively prime, then $\Delta(R) = 1$.
- If $b = p_1^{e_1} p_2^{e_2} p_3^{e_3} p_4^{e_4}$, $p_1, p_2, p_3, p_4 \equiv 3 \pmod{4}$, and $\phi(p_1^{e_1})/2, \phi(p_2^{e_2})/2, \phi(p_3^{e_3})/2, \phi(p_4^{e_4})/2$ are pairwise relatively prime, then $\Delta(R) = 0$.
- If $n = 8b$ where b is odd, then using similar logic as the previous case we have $\Delta(\mathbb{Z}/n\mathbb{Z}) = 1$ if and only if

$$D(U(\mathbb{Z}/n\mathbb{Z})) = D(U(\mathbb{Z}/b\mathbb{Z})) + 2.$$

Thus we can calculate $\Delta(R)$ in the following special cases:

- If b is a prime power then $\Delta(R) = 1$.
- If b is a product of distinct primes of the form $2^{2^n} + 1$, then $\Delta(R) = 1$.
- If $b = p_1^{e_1} p_2^{e_2}$ and $\gcd(\phi(p_1^{e_1}), \phi(p_2^{e_2})) = 1$, then $\Delta(R) = 1$.
- If $b = p_1^{e_1} p_2^{e_2} p_3^{e_3}$, $p_1, p_2, p_3 \equiv 3 \pmod{4}$, and

$$\gcd[\phi(p_1^{e_1})/2, \phi(p_2^{e_2})/2, \phi(p_3^{e_3})/2] = 2$$

and for $i \neq j$, $\gcd[\phi(p_i^{e_i})/2, \phi(p_j^{e_j})/2]$ has no prime factors greater than 7, then $\Delta(R) = 0$.

- If $b = p_1^{e_1} p_2^{e_2} p_3^{e_3} p_4^{e_4}$, $p_1, p_2, p_3, p_4 \equiv 3 \pmod{4}$, and $\phi(p_1^{e_1})/2, \phi(p_2^{e_2})/2, \phi(p_3^{e_3})/2, \phi(p_4^{e_4})/2$ are pairwise relatively prime, then $\Delta(R) = 0$.

However, the general cases $n = 4b, 8b$ remain open.

12.2. $R = \mathbb{F}_2[x]/(f)$. In this case, Corollary 3.2 gives the bound $\Delta(\mathbb{F}_2[x]/(f)) \leq 2$.

Question. For $f \in \mathbb{F}_2[x]$, when is $\Delta(\mathbb{F}_2[x]/(f))$ equal to 0, when is it 1, and when is it 2?

Again, we have already answered this question for most $f \in \mathbb{F}_2[x]$. Let $f = x^a(x+1)^b g$, where $\gcd(g, x(x+1)) = 1$. Then

- If $g = 1$, then $\Delta(R) = \delta_{a1} + \delta_{a2} + \delta_{b1} + \delta_{b2}$.
- If $a, b \neq 2$, then $\Delta(R) = \delta_{a1} + \delta_{b1}$.
- If $a = 2$ and $b \leq 1$ and $|U(\mathbb{F}_2[x]/(g))|$ is odd, then $\Delta(R) = b$. (Note that this condition corresponds to g being a product of distinct irreducibles.)
- If $a = b = 2$ and $U(\mathbb{F}_2[x]/(g))$ is cyclic (i.e. product of distinct irreducibles with pairwise relatively prime degree), then $\Delta(R) = 1$.
- If $a = 2$, $b \in [3, 14] \cup [17, 24] \cup [33, 40]$ and $U(\mathbb{F}_2[x]/(g))$ is cyclic, then $\Delta(R) = 0$.

12.3. **Recap.** For general $n \in \mathbb{Z}$, the rank of $U(\mathbb{Z}/n\mathbb{Z})$ is equal to the number of prime factors of n , which gets arbitrarily large.

Unfortunately, very little is known about $D(G) - M(G)$ for general G of large rank.

In a similar vein, the rank of $\mathbb{F}_2[x]/(f)$ for general $f \in \mathbb{F}_2[x]$ can get annoyingly large, and even the special case $\Delta(\mathbb{F}_2[x]/(x^2 g))$ with $g \in \mathbb{F}_2[x]$ irreducible is tricky to calculate.

For example, [5] gives for any $k > 0$, $D(C_{2k+1} \times C_2^r) = 4k + 1 + r$ for $1 \leq r \leq 4$ but $D(C_{2k+1} \times C_2^5) > 4k + 7$.

If f is an irreducible polynomial in $\mathbb{F}_2[x]$ of degree d , one can check that $U(\mathbb{F}_2[x]/(f^2)) \simeq C_{2^{d-1}} \times C_2^d$. Thus if $R_d = \mathbb{F}_2[x]/(x^2 f_d^2)$, where f_d is an irreducible polynomial of degree d , applying Theorem 3.2 gives

$$D(R_d) = \max(D(C_{2^{d-1}} \times C_2^d) + 2, D(C_{2^{d-1}} \times C_2^{d+1}))$$

which gives $\Delta(R_2) = \Delta(R_3) = 1$ but $\Delta(R_4) = 0$. The general problem of determining $\Delta(\mathbb{F}_2[x]/(f))$ for non-squarefree $f \in \mathbb{F}_2[x]$ is still open for reasons such as this. For more information about the growth of $D(C_{2k+1} \times C_2^d)$, we refer the reader to [6].

12.4. **Infinite Semigroups.** Finally, we look at the case where S is an infinite commutative semigroup. By Theorem 3.1 and Lemma 5.1, when S is either

- a group, or
- the semigroup of a commutative ring R under multiplication,

then $D(S)$ being finite implies S is finite. However, there are semigroups S such that S is infinite but S is finite. Thus we have the following question:

Question. For what other families \mathcal{F} of (commutative, unital) semigroups is it true $S \in \mathcal{F}$ and $|S| = \infty$ implies $D(S) = \infty$?

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